6-vertex model, Lipschitz functions, proper colorings and other lattice models with hard constraints

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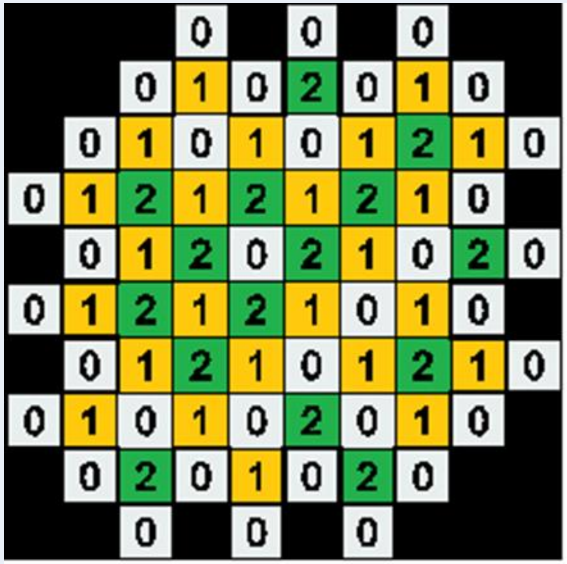
Proper colorings

- A proper q-coloring of $\mathbb{Z}^d$ is an assignment of colors $\{0,1,\ldots,q-1\}$ to the vertices with adjacent vertices colored differently.
- Put uniform distribution on proper colorings.
  This is exactly the zero-temperature antiferromagnetic q-state Potts model
- How does a typical sample look like?
  Is it disordered, critical or ordered?
- As a concrete question, when coloring a large cube, is the probability that two opposite corners are equally colored about $\frac{1}{q}$?



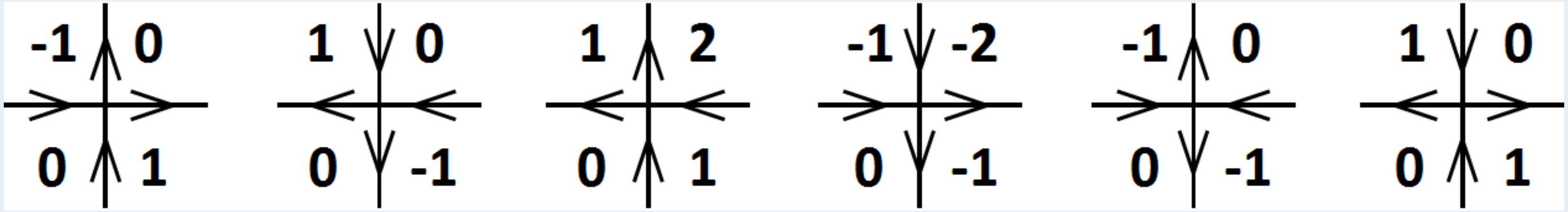
- The only parameters are q and d. No temperature parameter!
- Residual entropy – exponentially many microstates at zero temperature.
  Any emergent order is entropically driven.
  Lack of rigorous tools to establish such phenomenon.

First observations

- The model is ordered when $q = 2$: only configurations are the chessboard and its translation by one lattice unit.
- The model is disordered in $d = 1$ when $q \geq 3$.
- A precise phrasing of the question:
  What are the (periodic) ergodic Gibbs states of the model?
  Gibbs state – probability measure over colorings which satisfies that conditioned on the coloring outside each finite set, the coloring in the set is uniform on all possible extensions.
- Frozen configurations exist when $q \leq d + 1$.
  E.g., for $q = 3, \mathrm{d} = 2$:
  012012012012
  120120120120
  201201201201
- Thus focus on (periodic) ergodic Gibbs states of maximal entropy.

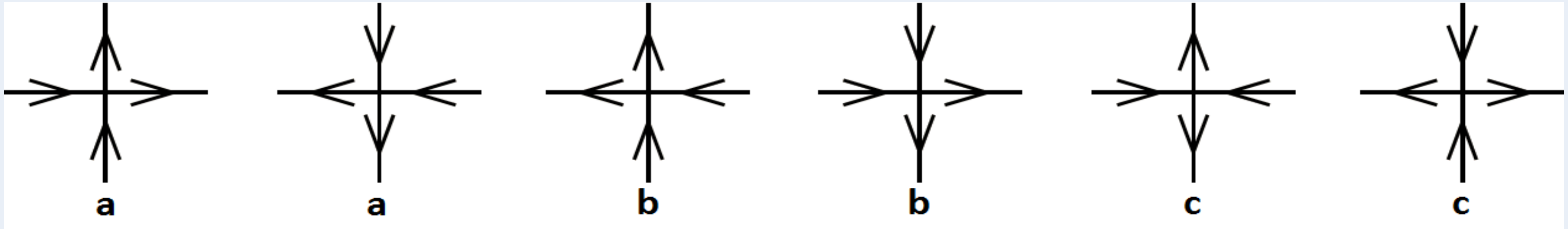
Height function representation for $q = 3$

- Proper 3-colorings admit a height representation:
  Consider functions $f: \mathbb{Z}^d \to \mathbb{Z}$ satisfying $|f(u) - f(v)| = 1$.
  Then f modulo 3 is a proper 3-coloring. In fact, this mapping is a bijection once the value of the coloring and the value of f are fixed at a single vertex.
- In $d = 1$ such functions are trajectories of simple random walk.
- In $d = 2$ such functions are height functions of the 6-vertex model, and specifically of square ice (uniform model).



6-vertex model

- Direct edges of square lattice so that every vertex has in-degree and out-degree two.
- Symmetric case: weights invariant under reversal of all arrows.

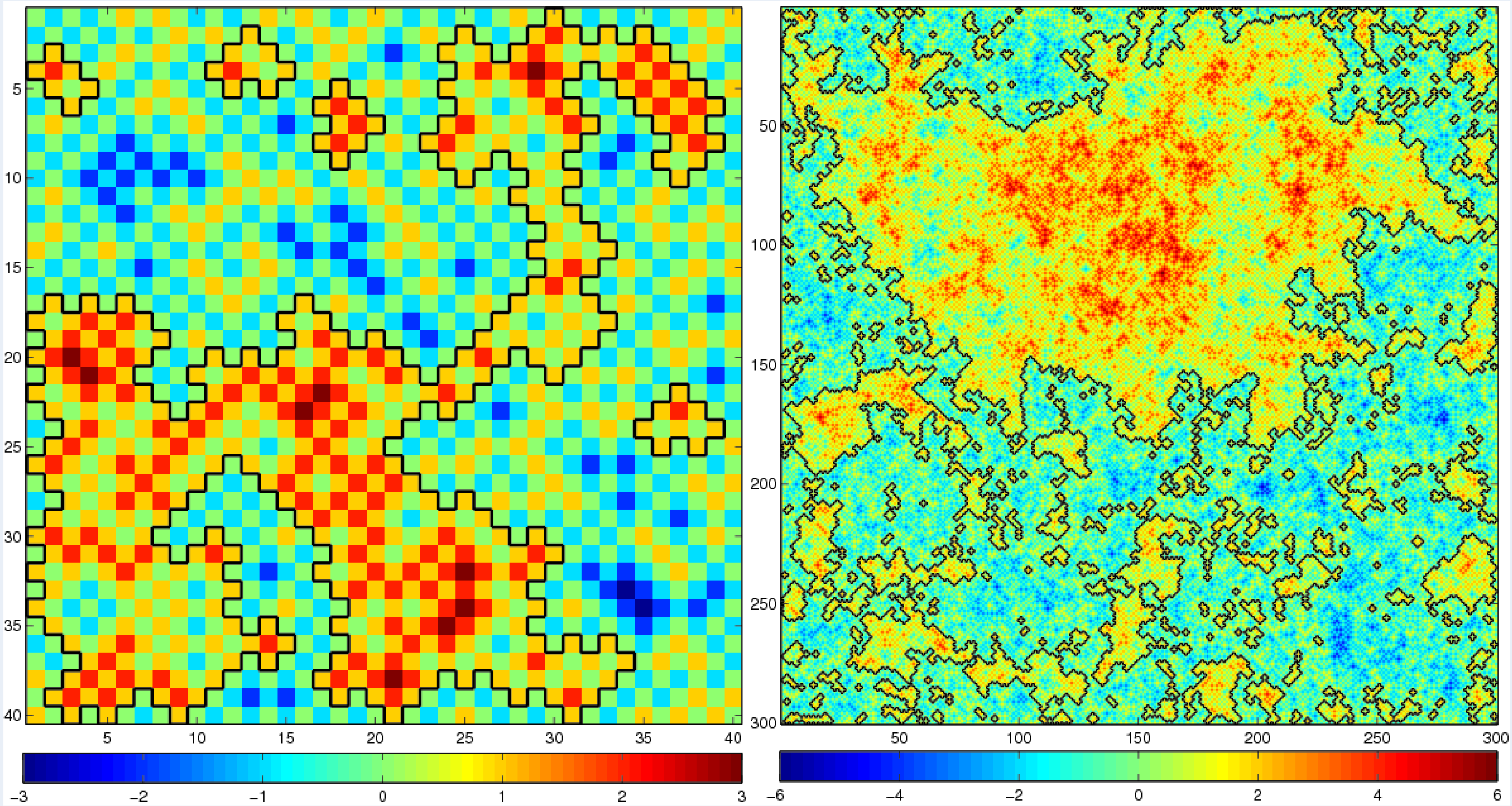


- Behavior determined by parameter (Lieb 1967, …)

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}$$

- $\Delta > 1$: Ferroellectric phase (stochastic 6-vertex model)
  $\Delta \in [-1,1)$: Disordered phase, includes free-fermion point $\Delta = 0$
  $\Delta < -1$: Anti-ferroelectric phase
- Uniform case $a = b = c$ ($\Delta = 1/2$) corresponds to square ice.

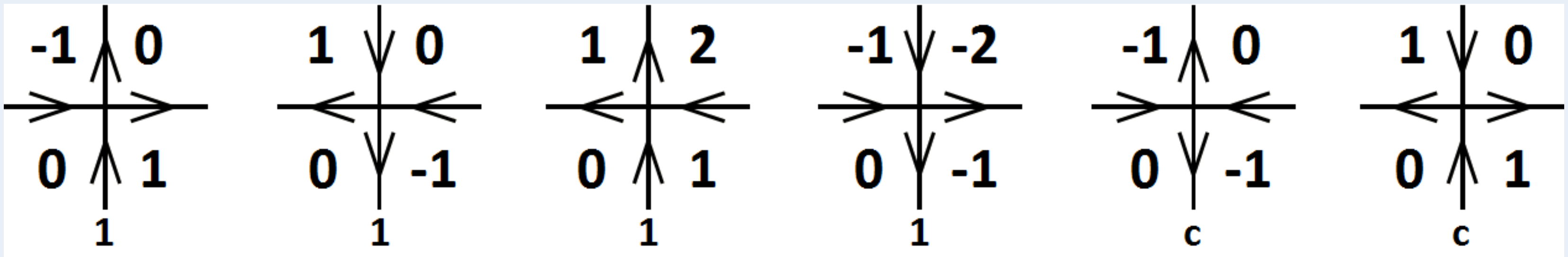
Square-ice height function (a=b=c=1)



Height function of square-ice with zero boundary values (on even vertices).
Non-trivial level sets separating heights 0 and 1 are highlighted.
Conjectured to scale to Gaussian free field, with level lines scaling to CLE(4).

Height function and F-model

- F-model: weights a=b=1 (no external magnetic field).



- Roughening transition:
  Weight c controls flatness of the height function.
  Rough for $0 < c \leq 2$ as $\Delta \in [-1,1)$. Flat for $c > 2$ as $\Delta < -1$.

- Mathematically difficult to establish! Especially hard to prove roughness.

- Predicts that proper 3-colorings in two dimensions are critical – exhibit power-law decay of correlations.

F-model – rigorous results

- Let f_L be the height function of the F-model on an $L \times L$ square with zero boundary conditions on even vertices.
- Measure roughness by variance at origin $\mathrm{Var}(f_L(0,0))$.
- Theorem (roughening transition, Glazman-P. 18):
  $c > 2$: Variance remains bounded as $L \to \infty$.
  $c = 2$: $\mathrm{Var}(f_L(0,0)) \asymp \log(L)$ as $L \to \infty$.
- Theorem (square-ice roughness, Chandgotia-P.-Sheffield-Tassy 18):
  $c = 1$: $\mathrm{Var}(f_L(0,0)) \to \infty$ as $L \to \infty$.
- Roughness + GFF scaling limit proved at free-fermion point $c = \sqrt{2}$ (Dubedat 11, Kenyon 00). Perturbative extension by Giuliani, Mastropietro, Toninelli (17).
- Roughness open for other values of c! No monotonicity result.

Square-ice – roughness proof (1)

- Normalize height functions to take even values on even sublattice.
- Goal: Show that there is no convergence in thermodynamic limit. Work with space $\mathcal{G}$ of Gibbs measures ergodic under parity-preserving translations. Assume $\mathcal{G} \neq \emptyset$ to obtain contradiction.
- Theorem (uniqueness up to additive constant, Sheffield 05, CPST): Let $\mu \in \mathcal{G}$ and f be sampled from μ. Then $\mathcal{G} = \{\text{distributions of } f + 2k \text{ for } k \text{ integer}\}$.
- Basic idea: Let $\mu_1, \mu_2 \in \mathcal{G}$. Sample f, g independently from μ_1, μ_2. Disagreement percolation (van den Berg 93, Sheffield 05): $P(f \neq g$ on infinite connected component$) = 0 \implies \mu_1 = \mu_2$. $P(f < g$ on infinite connected component$) = 0 \implies \mu_1 \succcurlyeq \mu_2$.
- Use percolation tools to study infinite connected component where $f \geq g + 2k$ for various values of k.

Square-ice – roughness proof (2)

- Previous technique applicable to height functions with general nearest-neighbor convex interaction. Now use idea specific to square-ice.
- Let $\mu \in \mathcal{G}$ and f be sampled from μ.
- Define function g by $g(v) = f\big(-v + (1{,}0)\big) - 1$ (reflect, shift, subtract one). Then the distribution of g is also in $\mathcal{G}$.
- Have $f(0{,}0) + f(1{,}0) = g(0{,}0) + g(1{,}0) + 2$.
- But theorem implies that $f = g + 2k$ in law, for some integer k.
- Thus $f(0{,}0) + f(1{,}0) = g(0{,}0) + g(1{,}0) + 4k$ in law.
- This is a contradiction, as $4k \neq 2$.

Back to proper colorings

- Recall our main question:
  To determine the behavior of proper q-colorings of $\mathbb{Z}^d$.
  So far: $\mathrm{d} \geq 1, q = 2$ – ordered
  $d = 1, q \geq 3$ – disordered
  $d = 2, q = 3$ – critical
- General principle: A model is disordered if the spin at a vertex is not significantly influenced by the spins at the rest of the configuration.
- Implies that coloring is disordered when q is large compared with d.
  E.g., when $q > 4d$ by the Dobrushin uniqueness theorem (68).
  In these cases, the AF Potts model is disordered at all temperatures.
- Is coloring ordered when d is large compared with q?
  In what ways can it order?

Ordering predictions

- Berker-Kadanoff (80) suggested power-law decay of correlations for fixed q in high dimensions, based on a simple renormalization consideration.
- Countered by numerical simulation and ε-expansion of Banavar-Grest-Jasnow (80) who predicted a Broken-Sublattice-Symmetry (BSS) phase for $q = 3,4$ in $d = 3$.
  Kotecký (85) also predicted BSS ordering for $q = 3$ in high dimensions by analyzing the model on a decorated lattice.
- Studied further by Engbers, Feldheim, Galvin, Kahn, P., Randall, Salas, Sokal, Sorkin, Spinka, Swart and others.

Ordering - rigorous result

- BSS State: Partition the q colors into subsets A, B of sizes $\left\lfloor\frac{q}{2}\right\rfloor, \left\lceil\frac{q}{2}\right\rceil$.
  An (A, B)-BSS state is a Gibbs state satisfying that a typical sample from it has most even vertices colored by colors from A and most odd vertices colored by colors from B (that is, a chessboard pattern of A and B colors).
  Breaks both color symmetry and lattice symmetry.
- Theorem (P.-Spinka 18): Suppose
  $$d \geq Cq^{10} \log^3 q\,.$$
  Then there exists an (A, B)-BSS state for each partition A, B of the q colors to subsets of sizes $\left\lfloor\frac{q}{2}\right\rfloor, \left\lceil\frac{q}{2}\right\rceil$.
  Moreover, every maximal-entropy periodic Gibbs state is a mixture of such states.
- Number of BSS states is $\binom{q}{q/2}$ for q even and $2\binom{q}{\lfloor q/2\rfloor}$ for q odd.

Remarks

- Extends earlier work showing existence of BSS states for $q = 3$ (P. 10, Galvin-Kahn-Randall-Sorkin 12).
  New for all $q \geq 4$.
- Result holds also in two dimensions for a modified lattice:
  $\mathbb{Z}^2 \times \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^d = \{(x_1, \dots, x_d): x_1, x_2 \in \mathbb{Z}, x_3, \dots, x_d \in \{0,1\}\}$.
- Extends to low-temperature antiferromagnetic Potts model (and other models, as we now explain).
- Emergent ordering is lattice-dependent.
  E.g., for each q may modify square lattice by replacing each edge by many parallel paths of length 2, to obtain lattice on which proper q-coloring orders by having a single color on most vertices of original square lattice (Huang et. al., PRE 2013).
  Qin et. al. (PRB 2014) find partial long-range order in certain cases.
  Our result is the first to treat the standard $\mathbb{Z}^d$ lattice.

General model and result

- General nearest-neighbor discrete spin model:
  Spins $[\mathrm{q}] = \{1, \ldots, q\}$.
  Interaction energies $(E_{a,b})$ for $a, b \in [q]$.
  Assume $E_{a,b} = E_{b,a}$. Allow $E_{a,b} = \infty$.
  Magnetic fields (E_a) for $a \in [q]$.
- Hamiltonian:
  $$\sum_{u \sim v} \sum_{a,b \in [q]} E_{a,b} \delta_{f(u),a} \delta_{f(v),b} + \sum_{v} \sum_{a \in [q]} E_a \, \delta_{f(v),a}$$
- No temperature parameter (included in specified energies).
- Examples: AF Potts model: $E_{a,b} = J$ if $a = b$, $E_{a,b} = 0$ if $a \neq b$.
  Lattice Widom-Rowlinson model, Lattice hard-core gas,
  clock models with hard constraints, ….

Dominant patterns and symmetry

- Assume WLOG that $\min E_{a,b} = 0$.
- Call $A, B \subset [q]$ a pattern if $E_{a,b} = 0$ for all $a \in A, b \in B$.
- The partition function of "pure (A, B)" configurations (configurations mapping even vertices to A and odd vertices to B) is exactly $(E_A E_B)^{\mathrm{volume}/2}$, where $E_A \coloneqq \sum_{a \in A} E_a$.
- Call a pattern (A, B) dominant if it maximizes $E_A E_B$ among all patterns.
- Symmetry assumption: Suppose that for every pair of dominant patterns $(A, B), (\mathrm{A}', \mathrm{B}')$ there exists a bijective map from $[q]$ to itself mapping one pattern to another and preserving all energies and magnetic fields.

BSS-type ordering

- Theorem (P.-Spinka 18): Assuming symmetry, there exists $d_0 = d_0((E_{a,b}), (E_a))$ such that in every dimension $d \geq d_0$: There is an (A, B)-BSS order for every dominant pattern (A, B). Moreover, every maximal-pressure periodic Gibbs state is a mixture of such states.

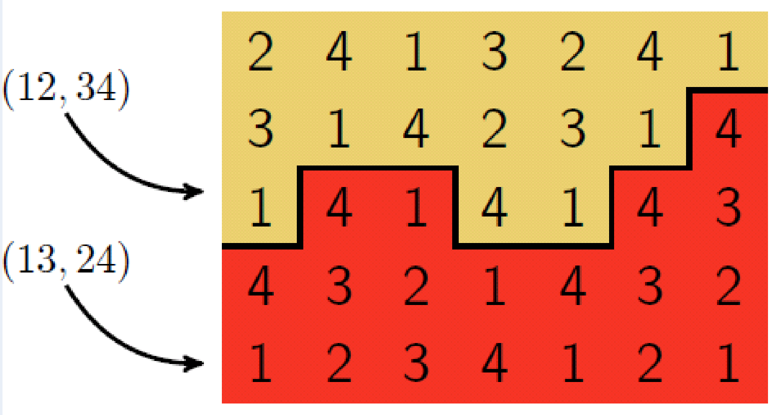
- Similar remarks as before apply. In particular, the theorem continues to be true when $\mathbb{Z}^d$ is replaced by $\mathbb{Z}^2 \times \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^d$.

- Quantitative bounds on d_0 are available. Depend on energy gaps and weight of "second-dominant pattern". Imply BSS ordering for AF q-state Potts model when d is large as a function of q and $\beta \geq d^{-\alpha}$ for some $\alpha > 0$.

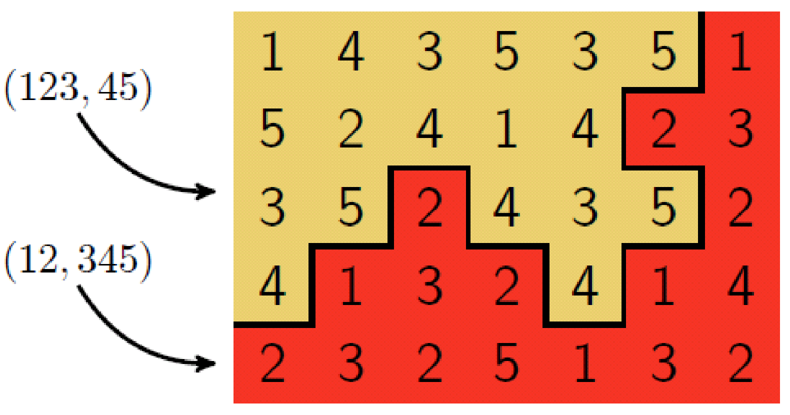
Ideas of proof for proper colorings

- Wish to implement a Peierls-type argument:
  1) Find ordered/disordered regions and domain walls in coloring.
  2) Show that the probability of any given set of contours being the domain walls is exponentially unlikely in their total length.
  3) Sum over contours to conclude that no long contours arise.
- Problems:
  1) How to define ordered regions?
  2) How to bound probability of a given set of contours (in absence of energy terms)?
  2) Too many contours to sum over.
- Solutions:
  1) Classify vertices to ordered regions by the colors of their neighbors.
  2) Use entropy inequality (Shearer's inequality) to bound probability of given contour picture.
  3) Note that relevant class of contours allows for coarse graining.

Ordered regions and domain walls



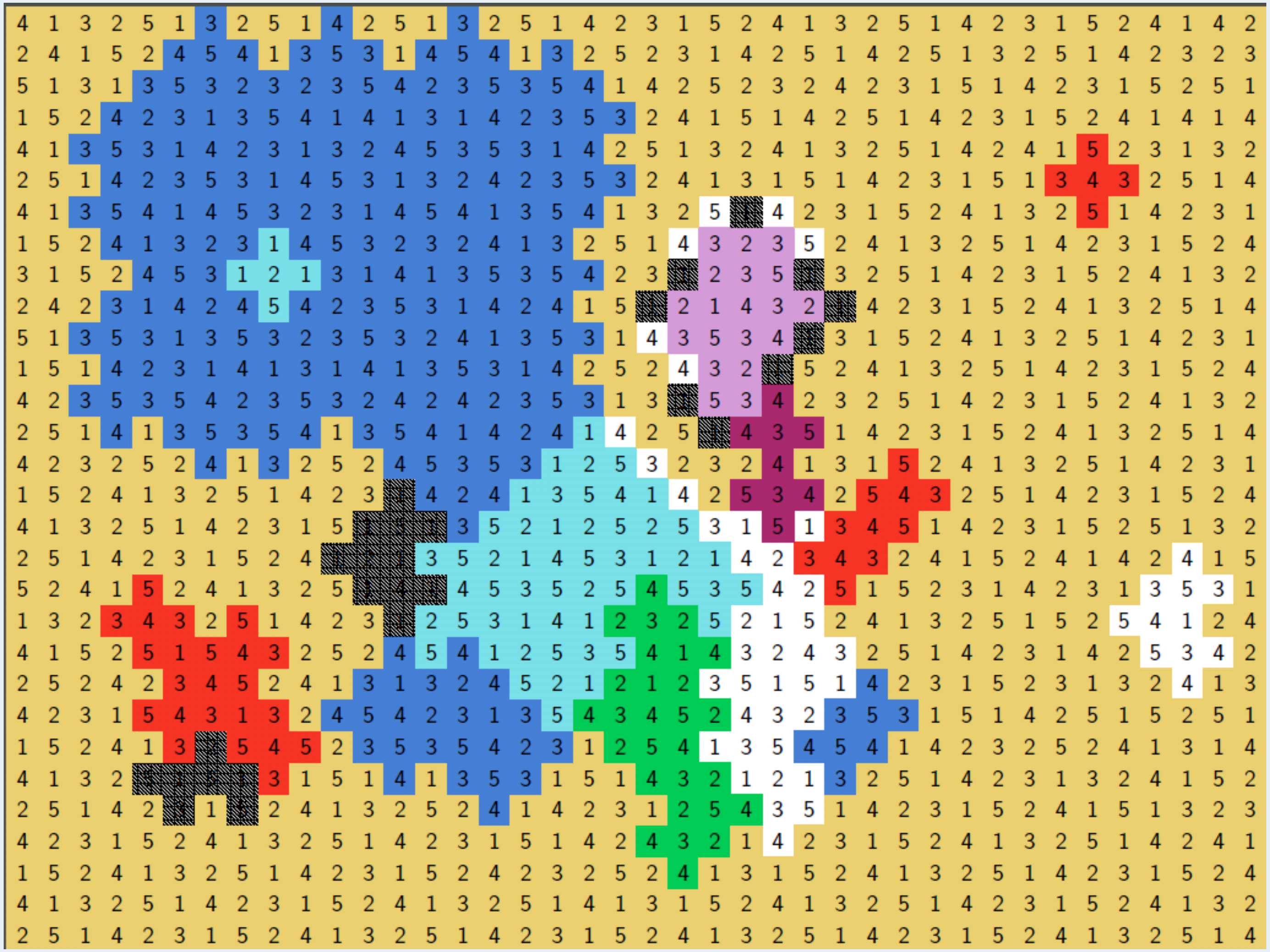
$q = 4$



$q = 5$

- Ordered regions may overlap.
- There are also disordered regions.
- For odd q the domain walls costing least entropy have all their inner boundary on one sublattice – such contours are termed odd cutsets. This is an essential feature for the coarse graining scheme used!

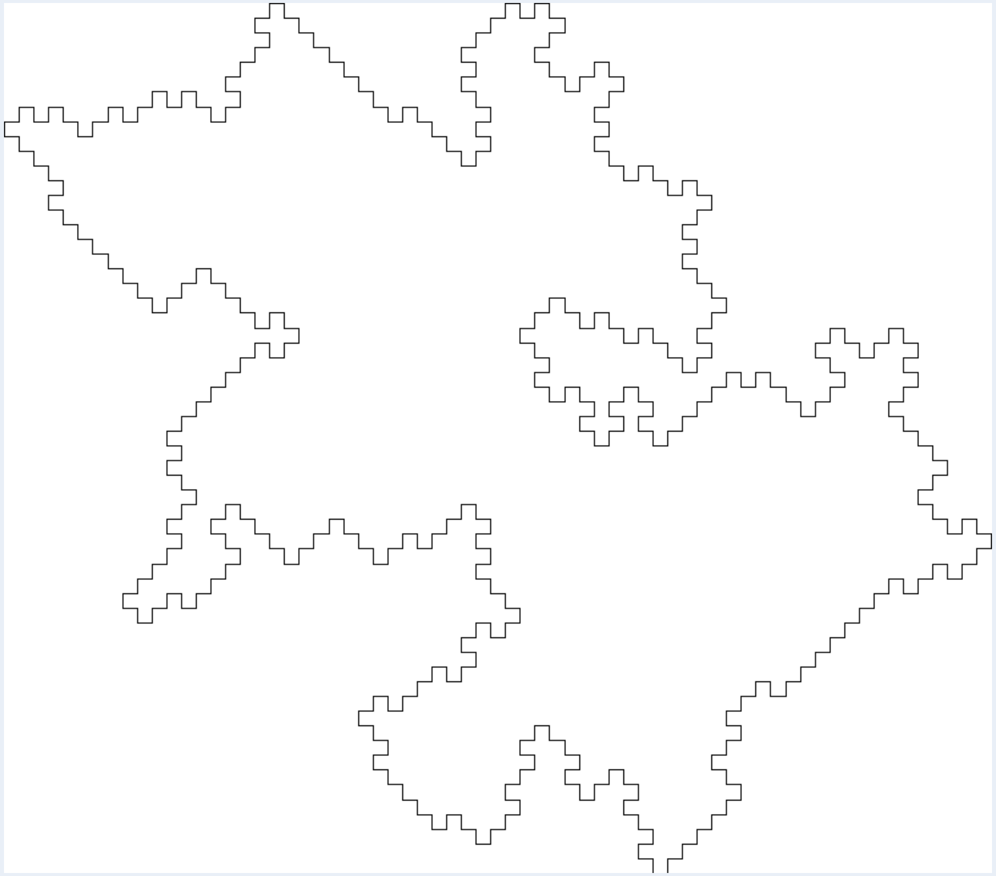
Classification to ordered/disordered regions



Colors – orderings. White – disordered. Black – overlap

Odd cutsets

- Contours having all their internal vertex boundary on same sublattice (used to study monotone Boolean functions by Korshunov 81, Sapozhenko 87).
- In high d, much fewer than standard contours: $2^{\left(\frac{1}{2d}+\varepsilon_d\right)L}$ versus. $d^{\frac{c}{d}L}$
  L – boundary length
  (Feldheim-Spinka 16, Lebowitz-Mazel 98).



- Breathing transition: most odd cutsets in high d seem to be small perturbations of boxes.
- Leads to efficient coarse graining scheme for odd cutsets in high d.

Disordered regions

- Use entropy inequality to show that disordered regions have less entropy.
- Shearer's inequality: $X_1, \dots, X_n$ random variables. $I_1, \dots, I_m \subset \{1, \dots, n\}$. Assume each index is contained in at least t subsets.

$$H(X_1, \dots, X_n) \le \frac{1}{t} \sum_{j=1}^{m} H\left((X_i)_{i \in I_j} \right).$$

- Use on coloring f by writing

$$H(f) = H(f_{even}) + H(f_{odd} | f_{even}) = H(f_{even}) + \sum_{v\ odd} H\left(f_v \middle| f_{N(v)}\right) \le$$

$$\sum_{v\ odd} \frac{1}{2d} H\left(f_{N(v)}\right) + H\left(f_v \middle| f_{N(v)}\right)$$

- Idea originates from Kahn-Lawrentz (99), Kahn (01), Galvin-Tetali (04).

Open questions

- Describe low-temperature behavior for all q, d. Understand dependence on lattice structure.
- Critical behavior of antiferromagnetic Potts model?
- Cases in which the symmetry assumption of the general model is violated? (entropic repulsion)
- Non-nearest-neighbor models? Models with directed edges?